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ON THE EIGENVALUES OF 2ND ORDER ELLIPTIC DIFFERENCE OPERATORS

by

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I. INTRODUCTION

Let L be a second order elliptic operator defined in a smooth domain $\Omega \subset \mathbb{R}^n$. That is

$$(1.1)\qquad Lu = -\sum \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + \sum b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

where, for every $\xi \in \mathbb{R}^n$ we have

$$(1.2a)\qquad \lambda_0 \sum \xi_j^2 \le \sum a_{ij}(x)\xi_i\xi_j \le \Lambda_0 \sum \xi_j^2 , \quad \forall x \in \Omega ,$$

for certain positive constants λ_0, Λ_0, with

$$(1.2b)\qquad 0 < \lambda_0 < \Lambda_0 .$$

We may assume the coefficients $a_{ij}(x), b_i(x), c(x) \in C^\infty(\bar{\Omega})$ and

$$(1.2c)\qquad c(x) \ge 0 .$$

Let N be a first order operator defined in Ω, i.e.

$$(1.3)\qquad Nu = \sum \beta_i(x) \frac{\partial u}{\partial x_i} + \alpha(x)u .$$

In this report we are concerned with the eigenvalue problem

$$(1.4a)\qquad Lu = \lambda Nu \quad \text{in } \Omega$$

$$(1.4b)\qquad u = 0 \quad \text{on } \partial\Omega$$

and the related discrete eigenvalue problem

$$(1.5a)\qquad L_h U_h = \lambda_h N_h U_h \quad \text{in } \Omega(h)$$

$$(1.5b)\qquad U_h = 0 \quad \text{on } \partial\Omega(h)$$

where L_h and N_h are finite-difference operators defined on a grid region $\Omega(h)$ with boundary $\partial\Omega(h)$.

In the case where L and N are self-adjoint, i.e.

$$(1.6) \qquad b_i(x) = 0 \ , \quad \beta_i(x) = 0 \ , \qquad i = 1,2,\ldots,n$$

and $\alpha(x)$ does not change sign in Ω while L_h and N_h correspond to positive definite matrices one can attack this problem via the appropriate variational principles (see Weinberger [12]).

In the general case if L_h and N_h arise from a Galerkin approach and one has general $L_2(\Omega)$ convergence results one may apply the results of Vainikko [10] and Osborn [7].

At first glance it would appear that one could somehow easily modify the finite-difference set-up so as to be able to apply the finite-element theory. This author was unable to do this.

The difficulty centers about the following points. Let

$$T = L^{-1}N \ , \quad T_h = L_h^{-1}N_h \ .$$

In order to apply the theory of [7] we require that

$$T_hf \to Tf \quad \forall f \in X$$

where X is an appropriate Banach space, say $L_2(\Omega)$. But, the finite-difference theory is not designed to operate on all of $L_2(\Omega)$, only on sufficiently smooth functions.

Moreover, if one attempts to extend the theory to all of $L_2(\Omega)$ via L_2 projection or "smoothing", one finds that the eigenvalue problem becomes perturbed in a manner which is non-trivial <u>when viewed from all of</u> $L_2(\Omega)$. On the other hand, a glance at the development of [7] shows that all of the "real action" takes place in certain smooth, finite dimensional subspaces; the eigenspaces. This remark is the key to the development given in this report.

This work is motivated by the desire to extend the theory of iterative methods for the solution of the algebraic problems associated with L_h developed in [8] to the non-self-adjoint case. See [9]. For this reason the operators N_h will approximate N in a relatively "weak" manner.

In order to simplify the presentation we first consider a particular "model" problem. Following that development we discuss the essential features of that discussion.

In section 2 we describe the basic model problem and some well-known facts about this model problem.

~~In section 3 we develop the convergence theorems.~~

In section 4 we discuss the ideas in our convergence proofs and their possible extension to more general problems.

2. THE MODEL PROBLEM

Let

$$\Omega \equiv \{(x,y);\ 0 < x,\ y < 1\} \tag{2.1}$$

and let

$$Lu = -(u_{xx}+u_{yy}) + a(x,y)u_x + b(x,y)u_y \tag{2.2}$$

where $a, b \in C^\infty(\overline{\Omega})$ are nice functions. Let

$$Nu = \alpha(x,y)u_x + \beta(x,y)u_y + q(x,y)u \tag{2.3}$$

with $\alpha, \beta, q \in C^\infty(\overline{\Omega})$. For every $f \in L_2(\Omega)$ we let

$$Tf = u$$

be the solution of

$$Lu = Nf\ , \quad \text{in} \quad \Omega\ , \tag{2.4a}$$

$$u = 0\ , \quad \text{on} \quad \partial\Omega\ . \tag{2.4b}$$

Since N maps $L_2(\Omega)$ into $H_{-1}(\Omega)$ and the resulting u is in $\overset{\circ}{H}_1(\Omega)$ we see that T is a compact map whose spectrum $\sigma(T)$ consists only of eigenvalues and 0 . Thus a point $z \in ₵$, $z \neq 0$, is in the spectrum of T only if there is a $u \in L_2(\Omega)$ which satisfies

$$Tu = zu\ , \quad u \not\equiv 0\ . \tag{2.5}$$

We assume that 0 is not an eigenvalue of T .

Let P be a fixed positive integer and set

$$h = \frac{1}{P+1} .$$

Let

(2.6a) $$\Omega(h) \equiv \{(x_i, y_j) = (ih, jh);\ 1 \le i,j \le P\}$$

be the set of interior mesh points while

(2.6b) $$\partial\Omega(h) \equiv \{(x_i, y_j) = (ih, jh);\ i \text{ or } j = 0 \text{ or } P+1\}$$

is the set of boundary mesh points. A "grid vector" $\{U_{ij};\ 0 \le i,j \le P+1\}$ is a function defined on the total set of mesh points

$$\overline{\Omega}(h) = \Omega(h) \cup \partial\Omega(h) .$$

As usual, we define the discrete Laplace Operator by: for $i \le i,j \le P$

(2.7) $$[\Delta_h U]_{i,j} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j+1}}{h^2} .$$

Let

(2.8a) $$a_{ij} = a(x_i, y_i) , \quad b_{ij} = b(x_i, y_i) .$$

With this notation we now define the finite-difference operator L_h corresponding to L. For $1 \le i,j \le P$

(2.8) $$[L_h U]_{ij} = -[\Delta_h U]_{ij} + a_{ij} \frac{U_{i+1,j} - U_{i-1,j}}{2h} + b_{ij} \frac{U_{i,j+1} - U_{i,j-1}}{2h} .$$

Remark: While U is defined on the entire mesh region, the values $\Delta_h U$ and $L_h U$ are defined only on the interior mesh points, $\Omega(h)$.

We also define the usual difference operators

(2.9a) $$[\nabla_x U]_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{h}, \quad 1 \le j \le P,\ 0 \le i \le P$$

(2.9b) $$[\nabla_y U]_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{h}, \quad 0 \le j \le P,\ 1 \le i \le P .$$

Let U,V be grid vectors. We define an inner product

(2.10a) $$(U,V) = h^2 \sum_{i=1}^{P} \sum_{j=1}^{P} \overline{U}_{ij} V_{ij} .$$

Observe that this inner product involves only the interior mesh points, $\Omega(h)$. In the usual way we define the semi-norm

(2.10b) $$||U||_h^2 = (U,U)$$

We note that this semi-norm is a norm for those grid vectors U which vanish on the boundary, i.e.,

(2.11) $$U = 0 \quad \text{on} \quad \partial\Omega(h) .$$

We recall the following basic facts.

Lemma 2.1: Let U be a grid vector which satisfies (2.11). Then

(2.12) $$(-\Delta_h U, U) = (\nabla_x U, \nabla_x U) + (\nabla_y U, \nabla_y U) .$$

Proof: Apply summation by parts. See [1], [11].

Definition: Let $f(x,y) \in C(\overline{\Omega})$. Then $Q_h f$ is the grid vector which interpolates at the mesh points. That is, $[Q_h f]$ is the restriction of f to $\Omega(h)$;

(2.13) $$[Q_h f]_{ij} = f_{ij} = f(x_i, y_j) .$$

Definition: Let $f \in C^2(\overline{\Omega})$ and let $\omega_2(f;h)$ be the modulus of continuity of the second derivatives of f, i.e. Let $\phi(x,y)$ be any second derivative of f, $\phi = f_{xx}$ or $\phi = f_{yy}$ or $\phi = f_{xy}$. For all $(x,y) \in \overline{\Omega}$ and $(x',y') \in \overline{\Omega}$ with

$$|x-x'| \le h , \quad |y-y'| \le h ,$$

we have

$$\sup \{|\phi(x,y) - \phi(x',y')|; \phi = f_{xx}, f_{xy}, f_{yy}\} = w_2(f;h) . \tag{2.14}$$

Of course, $\omega_2(f,h) \to 0$ as $h \to 0$.

Definition: Let L^* be the formal adjoint of the operator L. That is

$$L^*U = -(U_{xx}+U_{yy}) - (au)_x - (bu)_y . \tag{2.15a}$$

Let L_h^* be the finite-difference operator defined by

$$[L_h^*U]_{ij} = -[\Delta_h U]_{ij} - \left[\frac{a_{i+1,j}U_{i+1,j}-a_{i-1,j}U_{i-1,j}}{2h}\right] - \left[\frac{b_{i,j+1}U_{i,j+1}-b_{i,j-1}U_{i,j-1}}{2h}\right] . \tag{2.15b}$$

Lemma 2.2: Let U,V be grid vectors which vanish on $\partial\Omega(h)$, i.e. both U and V satisfy (2.11). Then

$$(U,L_hV) = (L_h^*U,V) . \tag{2.16}$$

If $U \in C^2(\overline{\Omega})$ then there is a constant $K > 0$ such that

$$L_h Q_h U = Q_h L U + \tau \tag{2.17a}$$

$$L_h^* Q_h U = Q_h L^* U + \tau^* \tag{2.17b}$$

where

$$\|\tau\|_\infty + \|\tau^*\|_\infty \le K\omega_2(U,h) \; . \tag{2.17c}$$

Proof: The identity (2.16) follows from an elementary summation by parts. The estimate (2.17c) follows from Taylor's theorem. Of course, if $U \in C^4(\bar{\Omega})$ then the truncation error $|E| + |E^*|$ is of order h^2 .

Lemma 2.3: Let F be a grid vector and let

$$\|F\|_\infty = \max \{|F_{ij}|; \; 1 \le i,j \le P\} \; .$$

There are constants $K > 0$, $h_0 > 0$ such that, if

$$0 < h \le h_0$$

and U is the solution of

$$L_h U = F \quad \text{in} \quad \Omega(h) \; , \tag{2.18a}$$

$$U = 0 \quad \text{on} \quad \partial\Omega(h) \; , \tag{2.18b}$$

then

$$\|U\|_\infty \le K\|F\|_\infty \; . \tag{2.19}$$

Proof: This well-known result follows from the discrete maximum principle and lemma 2.2. See [11].

<u>Lemma 2.4</u>: Let U be a grid vector which vanishes on $\partial\Omega(h)$. Then

$$-(\Delta_h U,U) \leq |(U,L_h U)| + \|a\|_\infty \|U\|_h \cdot \|\nabla_x U\|_h + \|b\|_\infty \|U\|_h \cdot \|\nabla_y U\|_h . \tag{2.20}$$

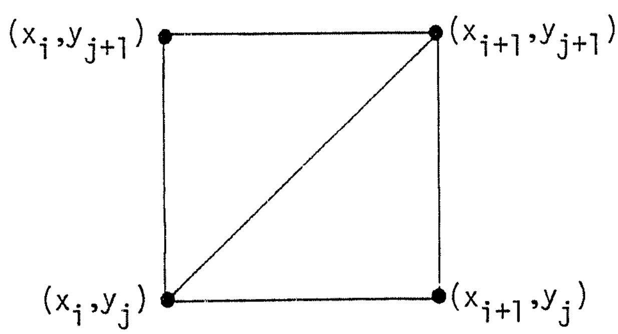
<u>Proof</u>: Direct Computation.

Before discussing the operators N_h we discuss the "prolongation" of grid vectors U into functions $U(x,y;h)$ defined on Ω.

Let the mesh region $\bar{\Omega}(h)$ be "triangulated". To be specific, imagine that the diagonal lines

$$y = x - jh , \quad j = -(P+1), -P,\ldots,P,(P+1)$$

have been drawn over the grid. Each square with vertices (x_i,y_j), (x_{i+1},y_j), (x_{i+1},y_{j+1}), (x_i,y_{j+1}) is considered as the union of two triangles.



With each grid vector $U = \{U_{ij}\}$ we associate a piecewise linear function $U(x,y;h)$ which is linear on each triangle and

$$U(x_i,y_j,h) = U_{ij} .$$

Let $P(h)$ be the space of all such function and $P_0(h)$ be the subspace of $P(h)$ which consists of those functions which vanish on $\partial\Omega$.

Throughout the remainder of this work we will make a complete identification between grid vectors $U = \{U_{ij}\}$ and these associated functions $U(x,y;h)$. This will avoid cumbersome notational difficulties.

Let $\langle\ ,\ \rangle$ be the $L_2(\Omega)$ inner product, i.e. if $f,g \in L_2(\Omega)$, then

(2.21) $$\langle f,g \rangle = \iint_{\Omega} \bar{f}(x,y)g(x,y)dxdy .$$

Lemma 2.5: Let $U,V \in P_0(h)$. Then

(2.22) $$\begin{aligned}\langle U,V \rangle = {} & \frac{10}{24}(U,V) + \\ & \frac{3h^2}{24} \sum_{i,j} \bar{U}_{ij}(V_{i+1,j}+V_{i-1,j}+V_{i,j+1}+V_{i,j-1}) \\ & + \frac{h^2}{24} \sum_{i,j} \bar{U}_{ij}(V_{i+1,j-1}+V_{i-1,j+1}) .\end{aligned}$$

Furthermore

(2.23a) $$\langle U,V \rangle = (U,V) + E_1$$

where

(2.23b) $$|E_1| \le h(U,U)^{\frac{1}{2}}(-\Delta_h V,V)^{\frac{1}{2}} .$$

Proof: The formula (2.22) is obtained by a direct integration. The estimates (2.23a) and (2.23b) follow from (2.22) and lemma 2.1.

Lemma 2.6: Let $U \in P(h)$. Let S_h be the triangle with vertices I,J,K. Then

$$(2.24) \qquad \iint_{S_h} U(x,y;h)\,dxdy = \frac{h^2}{6}(U_I+U_J+U_K) \ .$$

In this notation I,J and K are symbols for pairs of indices (i,j).

Proof: Direct integration.

We now turn to a discussion of the operator N_h . We do not describe these operators as specifically as we described L_h and L_h^* . Instead, we make the following weak consistency assumption.

Assumption N: Let $\overset{o}{N}_h$ be defined by: for $1 \leq i,j \leq P$

$$[\overset{o}{N}_h U]_{ij} = \alpha(x_i,y_j)\left[\frac{U_{i+1,j} - U_{i-1,j}}{2h}\right] + \beta(x_i,y_j)\left[\frac{U_{i,j+1} - U_{i,j-1}}{2h}\right]$$

$$+ q(x_i,y_j)U_{ij} \ .$$

We assume that, for all grid vectors U,V which vanish on $\partial\Omega(h)$,

$$(2.25a) \qquad (U,N_hV) = (U,\overset{o}{N}_hV) + E_2$$

where

$$(2.25b) \qquad |E_2| \leq Dh[(-\Delta_hU,U)\cdot(-\Delta_hV,V)]^{\frac{1}{2}}$$

$$+ Dh^2[(-\Delta_hU,U) + (-\Delta_hV,V)]$$

Lemma 2.7: Let $U \in P_0(h)$. Then there is a constant C_1 , depending on $||\alpha||_\infty$, $||\beta||_\infty$, $||q||_\infty$, but not on h so that

(2.26a) $$(\overset{o}{N}_h U, \overset{o}{N}_h U) \le C_1(-\Delta_h U, U) \; .$$

(2.26b) $$|(V, \overset{o}{N}_h U)| \le C_1 \|V\|_h (-\Delta_h U, U)^{\frac{1}{2}} \; .$$

Proof: Direct Computation.

Before proceeding to the convergence theorem of the next section, we recall some basic facts.

If $U \in P_0(h)$ then

(2.27) $$(-\Delta_h U, U) = \|U\|^2_{\overset{o}{H}_1(\Omega)}$$

Moreover, we have the important

Lemma 2.8 (Rellich): Let $h_n \to 0$ and let $U_n \in P_0(h_n)$ satisfy

$$(-\Delta_{h_n} U_n, U_n) = \|U_n\|^2_{\overset{o}{H}_1(\Omega)} \le K \; .$$

Then there is a subsequence n' and a function $u \in \overset{o}{H}_1(\Omega)$ such that

$$U_{n'} \rightharpoonup u \quad \text{in} \quad \overset{o}{H}_1(\Omega)$$

$$U_{n'} \to u \quad \text{in} \quad L_2(\Omega) \; .$$

Proof: See [2].

3. THE CONVERGENCE THEOREMS

Lemma 3.1: Let $\sum \subset \mathbb{C}$ be a compact subset of the resolvent set of T. Assume that

$$0 \notin \sum .$$

For each $z \in \sum$ consider the problem

$$L_h U - \frac{1}{z} N_h U = F , \quad \text{in } \Omega(h) , \tag{3.1a}$$

$$U = 0 , \quad \text{on } \partial\Omega(h) . \tag{3.1b}$$

There are constants $h_1 > 0$, $\bar{K} > 0$ with $\bar{K}$ independent of h, such that: For $0 < h \leq h_1$, problem 3.1 has a unique solution U, and

$$\|U\|_h \leq \bar{K}\|F\|_h . \tag{3.2}$$

Moreover, for every $u \in C^2(\overline{\Omega})$ which vanishes on $\partial\Omega$, let V be the grid vector which satisfies

$$L_h V - \frac{1}{z} N_h V = Q_h[Lu - \frac{1}{z} Nu] , \quad \text{in } \Omega(h) \tag{3.3a}$$

$$V = 0 , \quad \text{on } \partial\Omega(h) . \tag{3.3b}$$

Then

$$\langle V-u, V-u \rangle \to 0 \quad \text{as} \quad h \to 0 .$$

That is, $V(x,y;h)$ converges to u in $L_2(\Omega)$.

Proof: It is only necessary to verify (3.2). Suppose (3.2) is false. Then there are sequences $z_n \in \sum$, $h_n \to 0$ and grid vectors F_n such that

$$\|F_n\|_{h_n} \to 0 \quad \text{as} \quad h_n \to 0 \; , \tag{3.4a}$$

while the corresponding solutions [(3.1a), (3.1b], V_n, satisfy

$$\|V_n\|_{h_n} = 1 \; . \tag{3.4b}$$

Then

$$L_{h_n} V_n - \frac{1}{z_n} N_h V_n = f_n \; , \quad \text{in} \quad \Omega(h_n) \; , \tag{3.5a}$$

$$V_n = 0 \; , \quad \text{on} \quad \partial\Omega(h_n) \; . \tag{3.5b}$$

$$\|V_n\|_{h_n} = 1 \; . \tag{3.5c}$$

Applying lemma 2.4 we have

$$\begin{aligned} -(\Delta_{h_n} V_n, V_n) \le |(V_n, f_n)| + \|a\|_\infty \|\nabla_x V_n\|_{h_n} + \|b\|_\infty \|\nabla_y V_n\|_{h_n} \\ + \frac{1}{|z_n|} |(V_n, N_{h_n} V_n)| \; . \end{aligned} \tag{3.6}$$

Using Assumption N and Lemma 2.7 we have

$$|(V_n, N_{h_n} V_n)| \le C_1 (-\Delta_{h_n} V_n, V_n)^{\frac{1}{2}} + 2Dh_n (-\Delta_h V_n, V_n)$$

Therefore

$$-(\Delta_{h_n} V_n, V_n) \le \|f_n\|_{h_n} + K_2 + (\tfrac{1}{4} + \bar{D} h_n)(-\Delta_{h_n} V_n, V_n) \tag{3.7}$$

where

$$K_2 = 10(\|a\|_\infty^2 + \|b\|_\infty^2 + \bar{C}_1^2)$$

$$\bar{C}_1 = \max \{\frac{C_1}{|z|} , z \in \textstyle\sum\} .$$

$$\bar{D} = \max \{ \frac{2D}{|z|} , z \in \textstyle\sum\} .$$

Thus, when

$$\bar{D}h_n \leq \frac{1}{4}$$

we obtain

(3.8) $$(-\Delta_{h_n} V_n, V_n) \leq 2(1 + K_2) .$$

Let $z_{n'}$ be a subsequence of the z_n such that

$$z_{n'} \to \zeta \in \textstyle\sum .$$

Applying Lemma 2.8 we see that there is a sub-subsequence n'' and a function $v \in \overset{o}{H}_1$ such that

$$\text{(i)} \quad V_{n''} \rightharpoonup v \quad \text{in} \quad \overset{o}{H}_1(\Omega)$$

and

$$\text{(ii)} \quad V_{n''} \to v \quad \text{in} \quad L_2(\Omega) .$$

For simplicity of notation we denote $\{h_{n''}\}$ by $\{h_n\}$.

Let $\phi \in C_0^\infty(\Omega)$ and let

(3.9) $$\phi_n = Q_{h_n}\phi .$$

From (3.5a), (3.5b) we obtain

$$(\phi_n, L_{h_n} V_n) - \frac{1}{z_n}(\phi_n, N_h V_n) = (\phi_n, f_n) .$$

Thus, using Assumption N we see that

$$(L^*_{h_n}\phi_n, V_n) - \frac{1}{z_n}(\phi_n, \overset{o}{N}_h V_n) = (\phi_n, g_n)$$

where

$$\|g_n\|_{h_n} \to 0 \quad \text{as} \quad n \to \infty .$$

On letting $n \to \infty$ lemma 2.2 and lemma 2.5 yield

$$(3.10) \qquad \langle L^*\phi, v\rangle - \frac{1}{\zeta}\langle \phi, Nv\rangle = 0 .$$

Thus, v is a weak solution of

$$(3.11a) \qquad Lv - \frac{1}{\zeta}Nv = 0 , \quad \text{in} \quad \Omega ,$$

$$(3.11b) \qquad v = 0 , \quad \text{on} \quad \partial\Omega .$$

But, then, v is a genuine solution of (3.11a), (3.11b) and $\zeta \in \sum$ is an eigenvalue of T (see [2]). With this contradiction we have proven (3.2).

The remainder of the lemma follows from similar arguments.

Let $\mu \neq 0$ be an eigenvalue of T. Let

$$S \equiv \ker (\mu - T)^{\alpha} ,$$

i.e., the kernel or nullspace of the linear map $(\mu-T)^{\alpha}$ where α is the ascent or rank of μ (see [5]). S is the range of the linear projection E given by

(3.12) $$E = \frac{1}{2\pi i} \int_{\Gamma} (z-T)^{-1} dz$$

with Γ any circle in the resolvent set of T enclosing μ in its interior, but no other point of the spectrum. The linear projector E can also be written in the form

(3.13a) $$Ey = \sum_{i=1}^{a} \langle y,\phi_i^* \rangle \phi_i$$

where $\{\phi_i, i=1,2,\ldots a\}$ is any basis for S and $\{\phi_i^*, i=1,2\ldots a\}$ is the corresponding <u>dual basis</u> in

(3.13b) $$S^* = \ker (\mu - T^*)^{\alpha} .$$

That is $\phi_i^* \in S^*$ and $\langle \phi_i,\phi_j^* \rangle = \delta_{ij}$. Here T^* is determined by

(3.13c) $$\langle Tu,v \rangle = \langle u,T^*v \rangle .$$

That is,

(3.13d) $$T^* = N^*(L^*)^{-1} .$$

Let $\sum = \Gamma$. Applying Lemma 3.1 we see that

(3.14) $$(z-T_h)^{-1} = \frac{1}{z}(L_h - \frac{1}{z}N_h)^{-1} L_h$$

is well-defined on Γ for $0 < h \leq h_1$. Let

(3.15) $$E_h = \frac{1}{2\pi_i} \int_{\Gamma} (z-T_h)^{-1} dz .$$

<u>Lemma 3.2</u>: Let $\mu \neq 0$ be an eigenvalue of T. There is an $h_2 > 0$ such that if $0 < h \leq h_2$ there is an eigenvalue μ_h of T_h which lies in the interior of Γ.

<u>Proof</u>: If the lemma is false, then

$$E_h \equiv 0 .$$

Let $\phi \in S$ be an eigenfunction of T with eigenvalue μ. It is not difficult to prove that $\phi \in C^2(\overline{\Omega})$. However, in order to see how the general argument goes, we replace ϕ by a smooth $H_2 \cap \overset{o}{H}_1$ approximant, which we again call ϕ. Let

$$\phi_h = Q_h \phi , \quad \psi_h = E_h \phi_h \tag{3.16a}$$

and let

$$\Psi_h = Q_h E \phi , \tag{3.16b}$$

we observe that

$$\Psi_h(x,y;h) \to E\phi \neq 0 .$$

We calculate

$$\Psi_h - \psi_h = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z} [Q_h (L - \frac{1}{z} N)^{-1} L\phi - (L_h - \frac{1}{z} N_h)^{-1} L_h \phi_h] .$$

Since ϕ is smooth we may apply lemma 2.2. Thus

$$L_h \phi_h = Q_h L \phi + \tau$$

where $\|\tau\|_\infty \to 0$. In fact, for each fixed $z \in \Gamma$ let $v(x,y;z)$ be the solution of

(3.17a) $$(L - \frac{1}{z}N)v = L\phi \ , \quad v = 0 \text{ on } \partial\Omega \ ,$$

while V is the solution of

(3.17b) $$(L_h - \frac{1}{z}N_h)V = Q_h L_h \phi \ , \quad V = 0 \text{ on } \partial\Omega(h) \ .$$

From Lemma 3.1 we see that

$$\|V\|_h + (-\Delta_h V, V)^{\frac{1}{2}} \le K\|L\phi\|_\infty \ .$$

Thus, using Lemma 2.5 and Hölder's inequality we see that there is a constant K_1 such that

(3.18) $$\|V\|_{L_2(\Omega)} \le K_1 \ , \quad \|V\|_{L_1(\Omega)} \le K_1 \ .$$

Thus

(3.19) $$u(x,y;h) - V(x,y;h) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z}\,(v(x,y;z) - V(x,y;z))dz \ .$$

Applying Lemma 3.1 we see that

$$\|u-V\|_{L^2(\Omega)} \le \frac{1}{2\pi}\int_0^{2\pi} \theta d\theta \ \|v(\cdot,\cdot;z) - V(\cdot,\cdot;z)\|_{L^2(\Omega)} \ .$$

However, according to Lemma 3.1 the integrand goes to 0 for every $z \in \Gamma$ and the estimates (3.18) allow us to apply the dominated convergence theorem. Thus

$$V(x;y;h) \to E\phi \neq 0 \ .$$

Corollary: Let S_h denote the range of E_h . Then, for $0 < h \leq h_2$ we have

$$\dim S_h \geq \dim S = a \ .$$

Proof: We need only apply the above argument to each ϕ_i in the basis of S , smoothed if necessary. The resulting grid vectors $E_h Q_h \phi_i$ will be linearly independent if h is small enough.

Lemma 3.3: $\dim S = \dim S_h$.

Proof: Let $\mu_h \neq 0$ and V_h be an eigen pair of T_h . That is,

$$L_h V_h = \frac{1}{\mu_h} N_h V_h \ .$$

Suppose $\mu_h \to \bar{\mu}$. Then, it follows from Lemma 3.1 that $\bar{\mu}$ is an eigenvalue of T .

In fact, let V_h satisfy

$$(\mu_h - T_h)^{r-1} V_h = W_h \neq 0$$

$$(\mu_h - T_h)^{r} V_h = 0 \ .$$

A modification of the argument of Lemma 3.1 shows that, after selection of a subsequence if necessary, there are functions $v, w \in L_2(\Omega)$ such that

$$V_h \to v \ , \quad W_h \to w \neq 0$$

and

$$(\bar{\mu} - T)^{r-1} v = w \neq 0$$

$$(\bar{\mu} - T)^{r} v = (\bar{\mu} - T) w = 0 \ .$$

Thus, a basis for S_h converges to a set of linearly independent functions in S . Therefore

$$\operatorname{Lim\,sup} \dim S_h \leq \dim S = a \ .$$

Thus, the lemma follows from the corollary to Lemma 3.2.

We collect our results in the following

Theorem 3.1: Let $\mu \neq 0$ be an eigenvalue of T with algebraic multiplicity a . Let Γ be a circle about μ which does not include any other point of the spectrum of T either in its interior or on its boundary. Then there is ah $\bar{h}_1 > 0$ such that if $0 < h \leq \bar{h}_1$

(i) there are no eigenvalues μ_h of T_h on Γ

(ii) there are eigenvalues $\mu_{1,h}, \mu_{2,h}, \ldots \mu_{r,h}$ of T_h which lie inside Γ .

(iii) Let $m_1, m_2, \ldots, m_r$ be the algebraic multiplicities of $\mu_{1,h}$, $\mu_{2,h}, \ldots \mu_{r,h}$ respectively.

Then

$$\sum_{s=1}^{r} m_j = a \ .$$

Furthermore, the mapping

$$J_h = E_h Q_h \big|_S$$

is one to one and onto from S to S_h . Similarly, let

$$\tilde{J}_h = E \big|_{S_h} \ .$$

Then $\tilde{J}_h$ is also one to one and onto from $\tilde{S}_h$ to S .

Once one has this result one can follow Kreiss [6], Osborn [7] or de boor and Swartz [3] to obtain Theorem 3.2: Let

$$\bar{\mu} = \frac{1}{a} \sum_{(j)} m_j \mu_{j,h} \ .$$

Then there is a constant K such that

$$|\bar{\mu}-\mu| \leq K \sup \left.(T-T_h)\right|_S \ .$$

Specifically, we have

Let ϕ_i, $i=1,2,\ldots a$ be a basis for S and $\phi_i^*, 1=1,2\ldots a$ be a basis for S^* . Then there is a constant K such that

$$|\bar{\mu}-\mu| \leq K \sum_{\sigma=1}^{a} \langle (T-T_h)\phi_\sigma, \phi_\sigma^* \rangle$$

Proof: See the development in [3].

4. THE GENERAL IDEA

While the discussion in section 3 is limited to the model problem described in section 2, one can see the essential points and describe the basic ideas which will enable one to discuss more general problems.

The basic tool is the relationship between the linear projectors E given by (3.12) and E_h given by (3.15).

A look at the proof of Lemma 3.2 shows that one need only apply E_h to "smooth" elements ϕ_h. However, even when we apply E_h to a smooth ϕ_h, we must have results such as those given in Lemma 3.1 .

Thus it is worthwhile to study the argument in Lemma 3.1. Almost any discussion of the operator L_h yields estimates such as those given in Lemma 2.2 (consistency) and Lemma 2.3 (convergence in the smooth case). However, the use of E and E_h requires estimates on the solutions V_h of the equations

$$zL_h V - N_h V = f_h , \quad z \in \Gamma .$$

We have essentially used estimates on L_h^{-1} to obtain these estimates. The basic facts are

<u>FACT 1</u>: If $\{V_h ; 0 < h \leq h_0\}$ is the set of all "functions" in $P(h)$ such that

$$||L_h V_h||_h \leq C , \quad 0 < h \leq h_0 ,$$

then the $\{V_h\}$ are sequentially compact.

<u>FACT 2</u>: If $f_h, V_h \in P(h)$ and

(4.1a) $$zL_hV_h - \overset{o}{N}_hV_h = f_h$$

(4.1b) $$f_h \to f \in C^2$$

(4.1c) $$V_h \to V$$

then, in an appropriate sense

(4.2) $$zLV - NV = f \ .$$

In the proof of Lemma 3.1 we first use these facts to establish (3.2). Then we use these facts and (3.2) to prove the convergence of the solutions of (3.3a), (3.3b).

Thus, in the more general case, i.e. general L, N and Ω, one would expect to prove analogous spectral convergence theorems if one can establish these two facts. And, our discussion shows that this is the case. The proof of these facts are, in general, not too difficult. We have chosen to use "energy estimates" to establish FACT 1 and Weyl's lemma (weak solutions are genuine solution) to establish FACT 2.

In any particular instance one has a variety of possible arguments which lead to such conclusions. For example, in the case discussed in this report, i.e. the model problem, one could use the estimates of A. Brandt [4] to establish FACT 1. In fact, in this case, one can use the Banach space $C(\Omega)$ rather than $L_2(\Omega)$ and rely completely on the estimates of [4]. We choose to use "energy estimates" because we feel that they are (i) weaker and (ii) (generally) easier to obtain.

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